Strange Metals and Black Holes: A Holographic Correspondence

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Physical Motivation

The Sachdev–Ye–Kitaev (SYK) model is a solvable large-N theory of strongly interacting fermions with random all-to-all quartic interactions. In the infrared it realizes a non-Fermi-liquid regime capturing key features associated with **strange-metal** behavior:

1. **Absence of quasiparticles** due to strong many-body scattering.
2. **Conserved** $U(1)$ **charge density** $\mathcal{Q}$, tunable by a chemical potential μ.
3. **Residual entropy** $S(T\to 0)$ in the $N\to\infty$ limit.

These properties make SYK a useful framework for exploring strange-metal physics and its connection to near-horizon AdS_2 dynamics.

SYK model [Complex fermion generalization]

Hamiltonian with random Gaussian couplings:

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell - \mu \sum_i c_i^\dagger c_i, \tag{1}$$

with

$$J_{ij;kl} = -J_{ji;kl}, \quad J_{ij;kl} = -J_{ij;lk}, \quad J_{ij;kl} = J^*_{kl;ij}, \quad \overline{|J_{ij;k\ell}|^2} = J^2. \tag{2}$$

$U(1)$ conserved charge density:

$$\mathcal{Q} = \frac{1}{N}\sum_i \langle c_i^\dagger c_i \rangle. \tag{3}$$

Path integral formalism

- **Hubbard–Stratonovich Trans- formation**: To obtain an exact solution we introduce the auxiliary fields:

$$Q_{ab}(\tau,\tau') = Q_{ba}(\tau',\tau), \quad P_{ab}(\tau,\tau') = P^\dagger_{ba}(\tau',\tau). \tag{4}$$

Then, we obtain a quadratic action in the fermionic fields:

$$\begin{aligned} S = &\sum_{ia} \int_0^{1/T} d\tau\, c_{ia}^\dagger(\tau) \left(\frac{\partial}{\partial\tau} - \mu\right) c_{ia}(\tau) \\ &+ \sum_{ab} \int_0^{1/T} d\tau\, d\tau' \left[\frac{N}{4J^2}\big|Q_{ab}(\tau,\tau')\big|^2 + \frac{N}{2} Q_{ab}(\tau.\tau')\big|P_{ab}(\tau,\tau')\big|^2\right] \\ &- \sum_{ab} \int_0^{1/T} d\tau\, d\tau' Q_{ab}(\tau,\tau')\, P_{ba}(\tau',\tau) \sum_i c_{ia}^\dagger(\tau)\, c_{ib}(\tau'). \end{aligned} \tag{5}$$

- **Saddle-Point Equations:** In the large-N limit:

$$P_{ab}(\tau,\tau') = \langle c_a^\dagger(\tau) c_b(\tau')\rangle, \qquad Q_{ab}(\tau,\tau') = J^2\big|P_{ab}(\tau,\tau')\big|^2. \tag{6}$$

For replica-diagonal solutions (since we do not expect spin glass solutions):

$$P_{ab}(\tau,\tau') = \delta_{ab}\, G(\tau'-\tau) \quad , \quad G(\tau_1-\tau_2) = -\frac{1}{N}\sum_i \langle T_\tau c_i(\tau_1) c_i^\dagger(\tau_2)\rangle. \tag{7}$$

Emergent CFT_1 and $U(1)$ Gauge Invariance

In the low-energy scaling limit $\omega, T \ll J$, the $i\omega_n$ term is irrelevant. Defining the shifted self-energy $\tilde{\Sigma}(i\omega_n) = \Sigma(i\omega_n) - \mu$, the Dyson equations become:

$$\int d\tau_2\, G(\tau_1,\tau_2)\tilde{\Sigma}(\tau_2,\tau_3) = -\delta(\tau_1-\tau_3), \tag{8}$$

$$\tilde{\Sigma}(\tau_1,\tau_2) = -J^2[G(\tau_1,\tau_2)]^2 G(\tau_2,\tau_1). \tag{9}$$

These are invariant under:

$$\tau = f(\sigma), \tag{10}$$

given that the green function and the self-energy change as

$$G(\tau_1,\tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4}\frac{g(\sigma_1)}{g(\sigma_2)} G(\sigma_1,\sigma_2), \tag{11}$$

$$\tilde{\Sigma}(\tau_1,\tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4}\frac{g(\sigma_1)}{g(\sigma_2)} \tilde{\Sigma}(\sigma_1,\sigma_2). \tag{12}$$

This corresponds to emergent CFT_1 symmetry (reparametrization f) and $U(1)$ gauge symmetry (local phase g).

Low-energy Green's Function

Using the conformal map $\tau = \tan(\pi T\sigma)/(\pi T)$:

$$G(\sigma) = -C\frac{e^{-2\pi\mathcal{E}T\sigma}}{\sqrt{1+e^{-4\pi\mathcal{E}}}}\left(\frac{T}{\sin(\pi T\sigma)}\right)^{1/2}, \quad 0<\sigma<\frac{1}{T}. \tag{13}$$

SYK Entropy at $T\to 0$

Large-frequency expansion:

$$G(i\omega_n) = \frac{1}{i\omega_n} - \frac{\mu}{(i\omega_n)^2} + \mathcal{O}(i\omega_n)^{-3} \quad , \quad \mu = -\partial_\tau G(0^+) - \partial_\tau G(\beta^-). \tag{14}$$

In terms of the spectral density ρ_g:

$$\mu = -2\pi\mathcal{E}T - \int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\frac{\Omega[\rho_g(\Omega) - e^{-2\pi\mathcal{E}}\rho_g(-\Omega)]}{1+e^{-\Omega/T}}. \tag{15}$$

At low T, ρ_g is particle–hole symmetric, hence

$$\left(\frac{\partial\mu}{\partial T}\right)_{\mathcal{Q}} = -2\pi\mathcal{E} \quad \xrightarrow[\text{Relation}]{\text{Maxwell}} \quad \left(\frac{\partial S_{\rm SYK}}{\partial\mathcal{Q}}\right)_T = 2\pi\mathcal{E}. \tag{16}$$

Einstein–Maxwell Charged–AdS Geometry I

Einstein–Maxwell action (with cosmological constant):

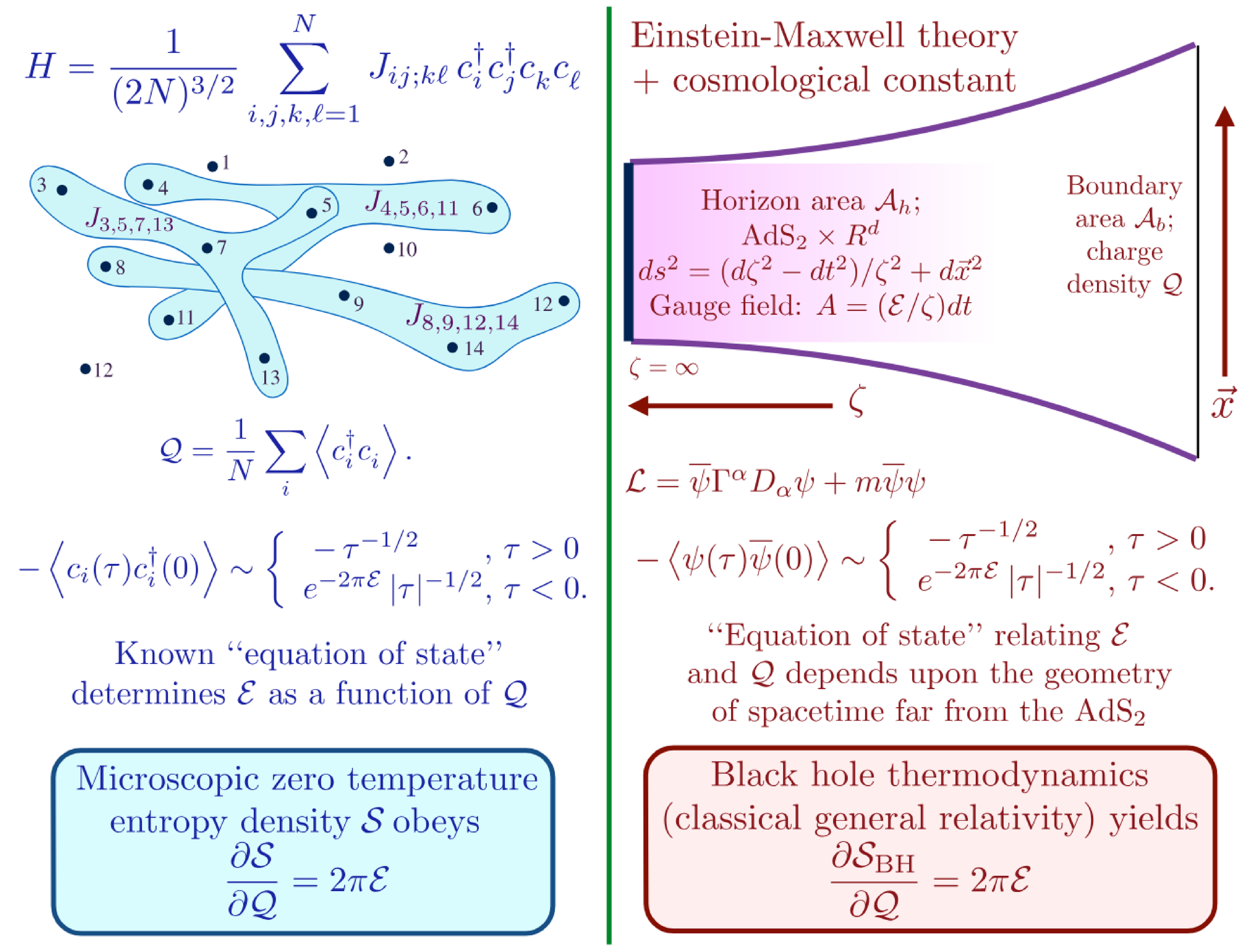
$$S = \frac{1}{2\kappa^2}\int d^{d+2}x\,\sqrt{-g}\left(\mathcal{R} + \frac{d(d+1)}{R^2} - \frac{R^2}{g_F^2}F_{\mu\nu}F^{\mu\nu}\right). \tag{17}$$

Solution:

$$ds^2 = \frac{r^2}{R^2}\big(-f(r)\,dt^2 + d\vec{x}^2\big) + \frac{R^2}{r^2 f(r)}dr^2, \tag{18}$$

where

$$f(r) = 1 + \frac{\Theta^2}{r^{2d}} - \left(r_0^{d+1} + \frac{\Theta^2}{r_0^{d-1}}\right)\frac{1}{r^{d+1}}. \tag{19}$$



Einstein–Maxwell Charged–AdS Geometry II

Black hole thermodynamic variables:

$$\mu = \sqrt{\frac{d}{2(d-1)}}\frac{g_F\Theta}{R^2 r_0^{d-1}}, \quad \mathcal{Q} = \sqrt{2d(d-1)}\frac{\Theta}{\kappa^2 R^d g_F}. \tag{20}$$

$$T = \frac{(d+1)r_0}{4\pi R^2}\left(1 - \frac{(d-1)\Theta^2}{(d+1)r_0^{2d}}\right), \quad S_{\rm BH} = \frac{2\pi}{\kappa^2}\left(\frac{r_0}{R}\right)^d. \tag{21}$$

Near-horizon geometry at $T\to 0$: $AdS_2\times\mathbb{R}^d$

At $T\to 0$, the near-horizon coordinate ζ is given by

$$r - \left(\frac{\Theta^2(d-1)}{d+1}\right)^{1/(2d)} = \frac{1}{\zeta}. \tag{22}$$

Near-horizon $AdS_2\times\mathbb{R}^d$ geometry:

$$ds^2 = R_2^2\frac{(-dt^2 + d\zeta^2)}{\zeta^2} + \frac{[\Theta^2(d-1)/(d+1)]^{1/d}}{R^2}d\vec{x}^2, \qquad R_2 = \frac{R}{\sqrt{d(d+1)}}. \tag{23}$$

Near-horizon entropy:

$$S_{\rm BH} = \frac{2\pi g_F|\mathcal{Q}|}{\sqrt{2d(d+1)}} = 2\pi\mathcal{Q}\mathcal{E}_{BH}, \quad \mathcal{E}_{BH} = \frac{g_F\,\mathrm{sgn}(\mathcal{Q})}{\sqrt{2d(d+1)}}. \tag{24}$$

AdS_2/CFT_1 correspondence

We can show that $\mathcal{E} = \mathcal{E}_{BH}$: The near-horizon geometry of the extremal Einstein–Maxwell RN–AdS black hole ($\mathbf{AdS}_2$) and the low-energy limit of the complex-fermion SYK model ($\mathbf{CFT}_1$) yield the same thermodynamic relation:

$$\frac{\partial S_{\rm BH}}{\partial\mathcal{Q}} = \frac{\partial S_{\rm SYK}}{\partial\mathcal{Q}}. \tag{25}$$

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Reference: S. Sachdev, *Bekenstein–Hawking Entropy and Strange Metals*, Phys. Rev. X **5**, 041025 (2015).