JOINT MARCH MEETING AND APRIL MEETING
Global Physics Summit

Path Integral Derivation of the Thermofield Double State in Causal Diamonds

(Class. Quantum Grav. 2025, 42 025015)

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Content

Introduction

1. What happened?

- **Hawking** (1974) [1]:

  Quantum fields + Black holes Geometry = B. H. Thermodynamics

$$T = \frac{\kappa}{2\pi} \quad , \quad S = \frac{A}{4}. \tag{1}$$

- The static observer has no access to the full geometry.

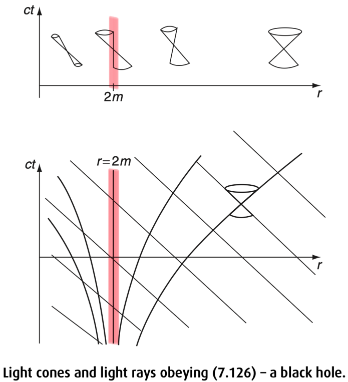


Figure 1: L. Ryder, 2009.

- **Unruh Effect** (1973-76) [2, 3, 4, 5, 6]:

  The vacuum state of a quantum field theory on Minkowski spacetime looks like a thermal equilibrium state for a uniformly accelerated observer with acceleration a.

$$T = \frac{a}{2\pi} \tag{2}$$

- The accelerated observer has no access to the full geometry.

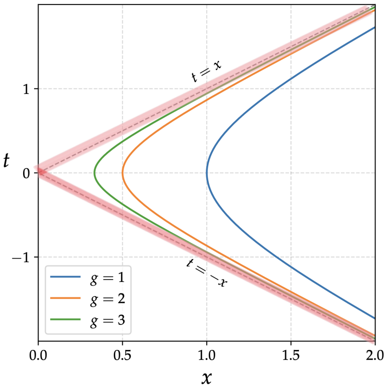


Figure 2: G. Valdivia-Mera, IJMP-D.

- Umezawa and Takahashi (1975) [7, 8, 9, 10]:

Introduce the TFD state $\Rightarrow$ Temperature dependent "vacuum state" $\Rightarrow$ Fictitious system.

$$\langle 0(\beta)| F |0(\beta)\rangle = Z^{-1}(\beta) \sum_n e^{-\beta E_n} \langle n| F |n\rangle \quad , \quad |0(\beta)\rangle = Z^{-1/2}(\beta) \sum_n e^{-\beta E_n/2} |n, \widetilde{n}\rangle \, . \tag{3}$$

Statistical average of an observable = expectation value

- Israel (1976) [11]: Thermofield dynamics of quantum fields in the presence of Black Holes:
    - $|0(\beta)\rangle$: Vacuum state of ϕ on the extended geometry is a thermal equilibrium state for the static observer.
    - Fictitious system: Defined on the time-reversed copy of the original spacetime in the extended geometry.

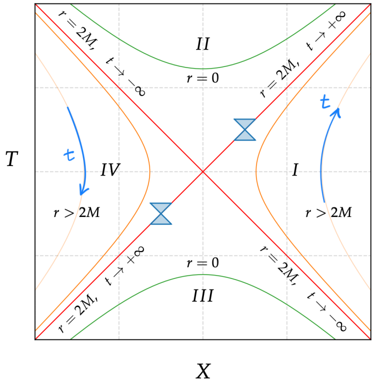


Figure 3: G. Valdivia-Mera, IJMP-D.

Causal horizons:

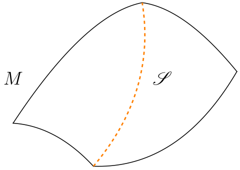
- The observers do not see the whole picture; they are constrained to regions of spacetime $\Rightarrow$ Particle modes are restricted due to the presence of horizons.
- Thermodynamic description of the systems due to quantum effects.

- Laflamme (1988) [12]:

Using the Euclidean path integral approach $\Rightarrow$ Systematically derive the TFD state.

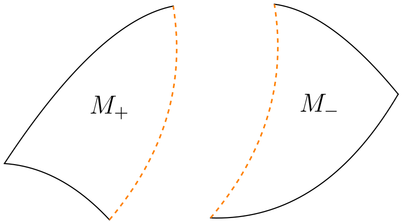
2. Our tools

2.1 Euclidean Path Integral for the scalar field Φ defined on the manifold M



$$P(A) = \int D\Phi \, \Pi(A) \, e^{-S_E[\Phi]}. \tag{4}$$

Requirement A: The existence of a surface $\mathscr{S}$ that divides M into two parts $M_\pm$: $\Pi(A) = \{0, 1\}$:



$$P(A) = \Psi_+(A)\Psi_-(A). \tag{5}$$

2.2 The manifold M is a cylinder

$$ds_E^2 = d\tau^2 + dx^2 \quad , \quad \tau \sim \tau + \beta \qquad (6)$$

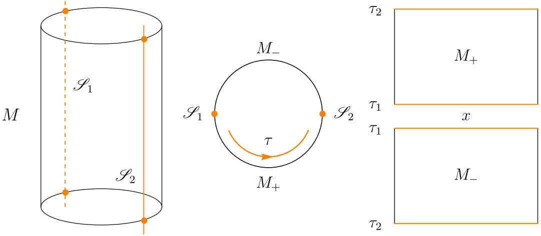


Figure 5: Boundary conditions:
$\tilde{\Phi} = \{\Phi_1, \Phi_2\}$ on $\mathscr{S} = \{\mathscr{S}_1, \mathscr{S}_2\}$.

Transition Amplitude

$$\Psi_+[\Phi_1, \Phi_2] = \underset{\mathscr{C}_+}{\int_{\Phi_1}^{\Phi_2}} D\Phi \, e^{-S_E[\Phi]} = \langle \Phi_2 | \, e^{-\frac{\beta}{2} H} \, | \Phi_1 \rangle \, , \quad (7)$$

$$\Psi_-[\Phi_1, \Phi_2] = \underset{\mathscr{C}_-}{\int_{\Phi_2}^{\Phi_1}} D\Phi \, e^{-S_E[\Phi]} = \langle \Phi_1 | \, e^{-\frac{\beta}{2} H} \, | \Phi_2 \rangle \, . \quad (8)$$

$$P_{\mathscr{S}}[\tilde{\Phi}] = \Psi_+[\Phi_1, \Phi_2] \Psi_-[\Phi_1, \Phi_2]. \qquad (9)$$

Strategy

Under certain transformations: $S[\phi, g_{\mu\nu}] \Rightarrow S[\phi]_{\text{SHO}}$.

The path integral can be computed exactly!

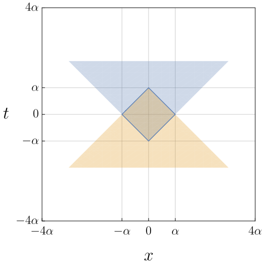
$$\Psi_\pm[\Phi_1, \Phi_2] = \frac{1}{\sqrt{Z(\beta)}} \sum_{n=0}^{\infty} e^{-\frac{\beta}{2} E_n} \varphi_n[\Phi_1] \varphi_n[\Phi_2]. \tag{10}$$

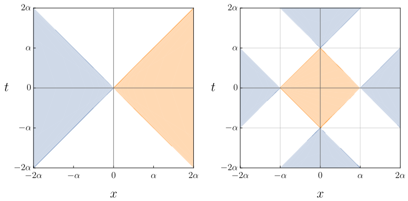
Density matrix of the physical system

$$\begin{aligned} \rho(\Phi_1, \Phi_1') &= \int d\Phi_2 \, \Psi_+[\Phi_1, \Phi_2] \, \Psi_-[\Phi_1', \Phi_2] \\ &= \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n} \varphi_n[\Phi_1] \varphi_n\left[\Phi_1'\right] . \end{aligned} \tag{11}$$

Thermofield Double State in causal diamonds

Causal diamond geometry





- Birth event $\cap$ Death event of an observer at $x = 0$.
- α: Lifetime parameter.
- Causal diamond boundary = Causal Horizons: Thermodynamics?

From [13]:

$$(t_d, x_d) = T(-\alpha) \circ K\left(\frac{1}{2\alpha}\right)(t_r, x_r) \ . \quad (12)$$

Lorentzian signature

$$
\begin{aligned}
t_r &= \xi \sinh\left(\frac{\eta}{\alpha}\right), \\
x_r &= \xi \cosh\left(\frac{\eta}{\alpha}\right).
\end{aligned} \tag{13}
$$

$$\Downarrow$$

$$
\begin{aligned}
t_d &= \frac{4\alpha^2 \xi \sinh\left(\frac{\eta}{\alpha}\right)}{4\alpha^2 + 4\alpha\xi \cosh\left(\frac{\eta}{\alpha}\right) + \xi^2}, \\
x_d &= \frac{\alpha\xi^2 - 4\alpha^3}{4\alpha^2 + 4\alpha\xi \cosh\left(\frac{\eta}{\alpha}\right) + \xi^2}.
\end{aligned} \tag{14}
$$

$$\Downarrow$$

$$ds^2 = \Omega^2(\eta, \xi)\left[-\frac{\xi^2}{\alpha^2} d\eta^2 + d\xi^2\right]. \tag{15}$$

Euclidean signature $(\eta = -i\tau)$

$$ds_E^2 = \Omega_E^2(\tau, \xi)\left[\frac{\xi^2}{\alpha^2} d\tau^2 + d\xi^2\right]. \tag{16}$$

Conformal factor

$$
\begin{aligned}
\Omega_E^2(\tau, \xi) &= \Omega^2(-i\tau, \xi) \\
&= \frac{16\alpha^4}{\left(4\alpha\xi \cos\left(\frac{\tau}{\alpha}\right) + \xi^2 + 4\alpha^2\right)^2}.
\end{aligned} \tag{17}
$$

Exact solution of the Euclidean path integral

Euclidean action

$$\textbf{CFT}: \quad S_E[\phi] = \frac{1}{2}\int \sqrt{g}\, d^2x g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{18}$$

Using $g_{\mu\nu}$ of Eq. (16) with $\xi = \alpha e^{\rho/\alpha}$, such that $-\infty < \rho < \infty$, we obtain:

Looks like the one in the Euclidean plane

$$S_E[\phi] = -\frac{1}{2}\int d\rho d\tau \phi\left[\partial_\tau^2 + \partial_\rho^2\right]\phi. \tag{19}$$

Conformal to the Euclidean plane:

$$ds_E^2 = \Omega_E'^2(\tau,\rho)(d\tau^2 + d\rho^2), \tag{20}$$

$$\Omega_E'^2(\tau,\rho) = \frac{16 e^{\frac{2\rho}{\alpha}}}{\left(4e^{\rho/\alpha}\cos\left(\frac{\tau}{\alpha}\right) + e^{\frac{2\rho}{\alpha}} + 4\right)^2}. \tag{21}$$

Important:

- The background geometry in the action (19) is conformal to the Euclidean plane with coordinates $\{\tau, \rho\}$.
- Introducing the periodicity of β in τ, ($\tau \sim \tau + \beta$), we obtain the cylindrical manifold M.
- We can divide the action (19) over the submanifolds $M_{\pm}$:

$$\{\mathscr{S}_1 = \tau_1, \mathscr{S}_2 = \tau_2\} \quad , \quad \tau_2 = \tau_1 + \beta/2. \tag{22}$$

$$0 < \tau_{M_+} < \beta/2 \quad , \quad \beta/2 < \tau_{M_-} < \beta. \tag{23}$$

Finding the SHO

- Fourier transform ϕ: $\rho \to \lambda$.
- Discretize the path integral for each mode λ_α.
- Build a real field ψ_{λ_α}.

Euclidean action for a single mode λ:

$$S_E[\psi_\lambda] = -\frac{1}{2}\int_{\mathscr{C}_\pm} d\tau \psi_\lambda(\tau)\left[\partial_\tau^2 - \lambda^2\right]\psi_\lambda(\tau). \tag{24}$$

Euclidean path integral for a single mode λ:

$$\Psi_\pm[\psi_1, \psi_2] = \int_{\mathscr{C}_\pm} D\psi_\lambda \; e^{-S_E[\psi_\lambda]}, \tag{25}$$

$\{\psi_1, \psi_2\}$ are the boundary configurations of the field ψ_λ on $\{\mathscr{S}_1 = \tau_1, \mathscr{S}_2 = \tau_2\}$.

Exact solution for the action (24):

$$\Psi_{+}[\psi_1,\psi_2] = \left(\frac{\lambda}{2\pi\sinh\left(\frac{\lambda\beta}{2}\right)}\right)^{1/2} \exp\left[-\frac{\lambda}{2}\left(\left(\psi_1^2+\psi_2^2\right)\coth\left(\frac{\lambda\beta}{2}\right) - \frac{2\psi_1\psi_2}{\sinh\left(\frac{\lambda\beta}{2}\right)}\right)\right]. \tag{26}$$

$$\Downarrow$$

Wave functional form of the TFD state $|0(\beta)\rangle$:

$$\Psi_{+}[\psi_1,\psi_2] = \frac{1}{\sqrt{Z(\beta)}}\sum_{n=0}^{\infty} e^{-\frac{\beta}{2}E_n}\varphi_n[\psi_1]\varphi_n[\psi_2], \tag{27}$$

where $\varphi_n[\psi_{1,2}]$ is the n-excited state of the harmonic oscillator applied to the boundary values of the field: $\varphi_n[\psi_{1,2}] = \langle\psi_{1,2}|n\rangle = \left(\sqrt{\frac{\lambda}{\pi}}\frac{1}{2^n n!}\right)^{1/2} H_n(\sqrt{\lambda}\psi_{1,2})e^{-\frac{\lambda}{2}\psi_{1,2}^2}$.

What is $\psi_{1,2}$ in the context of the causal diamond?

$$\psi_2 = \psi_{in} \quad , \quad \psi_1 = \tilde{\psi}_{in} = \Theta_D \psi_{ext} \Theta_D^{-1} \tag{28}$$

$$\Theta_D : \mathcal{H}_{ext} \to \mathcal{H}_{in} \quad , \quad \tilde{\psi}_{in} = \Theta_D \psi_{ext} \Theta_D^{-1} \quad , \quad |\tilde{\psi}_{in}\rangle = \Theta_D |\psi_{ext}\rangle \tag{29}$$

$$\Psi_+[\psi_{ext}, \psi_{in}] = \langle \psi_{in}| \, e^{-\frac{\beta}{2} H} \Theta_D |\psi_{ext}\rangle \propto \langle \psi_{ext}, \psi_{in} | 0(\beta)\rangle \, . \tag{30}$$

TFD state for the Causal Diamond

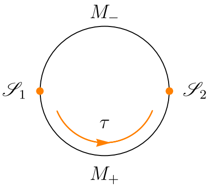
$$|0(\beta)\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_{n=0}^{\infty} e^{-\frac{\beta}{2} E_n} |n_{in}\rangle \otimes \Theta_D^{-1} |n_{in}\rangle \, . \tag{31}$$

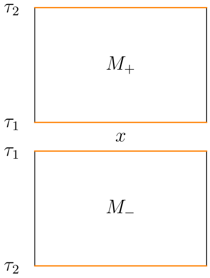
The observer constrained to the causal diamond, which is causally disconnected from the rest of spacetime, perceives the vacuum state as a thermal bath at temperature β^{-1}.

Diamond temperature for static observer

- $\{\mathscr{S}_1, \mathscr{S}_2\} \supset$ Birth and death event.
- $\{\tau = \pm\beta/2\} \iff \{t_E = \pm\alpha\}$.

$$\tau = \alpha \tan^{-1}\left(\frac{4\alpha^2 t_E}{4\alpha^2(x_d + \alpha) - 2\alpha[(x_d + \alpha)^2 + t_E^2]}\right) . \tag{32}$$





- For a static observer at $x = 0$:

$$\tau(t_E = -\alpha, x = 0) = -\pi\alpha/2,$$
$$\tau(t_E = \alpha, x = 0) = \pi\alpha/2. \tag{33}$$

Geometry - Thermodynamics - Lifetime

$$\beta = T^{-1} = \pi\alpha . \tag{34}$$

The density matrix of the physical system

Density matrix for the pure state given by the TFD state (31):

$$\rho = |0(\beta)\rangle \langle 0(\beta)| \, . \qquad (35)$$

Density matrix for the physical system ψ_{in}:

$$\rho_{in} = \mathrm{Tr}_{ext}\,\rho = \sum_m \langle \tilde{m}_{ext}|\,\rho\,|\tilde{m}_{ext}\rangle = \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n}\,|n_{in}\rangle\langle n_{in}| \qquad (36)$$

Density matrix of the physical system

(Canonical Emsemble)

$$\rho_{in} = \frac{e^{-\beta H}}{Z(\beta)} . \qquad (37)$$

Conclusions

- The "vacuum state" perceived by the observer confined to the interior region of the causal diamond is expressed through the density matrix ρ_{in}, which describes a mixed ensemble in thermal equilibrium with a heat bath at a temperature given by $\beta^{-1} = 1/\pi\alpha$.
- Finally, since ρ_{in} describes a mixed ensemble, the TFD state is an entangled state of the Hamiltonian eigenbasis $\{|n_{in}\rangle, |\tilde{n}_{ext}\rangle = \Theta_D^{-1}|n_{in}\rangle\}$.

Acknowledgements

- The author C.V-M. would specially like to thank the Center for Mexican American and Latino/a Studies at the University of Houston for their generous support with a Lydia Mendoza Fellowship.
- The authors C.R.O. and G.V-M. were partially supported by the Army Research Office (ARO) under Grant W911NF-23-1-0202.
- The author G.V-M. wishes to thank Nada Eissa for her assistance, which was of great worth in preparing these slides.

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